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ON THE ISOMETRIES OF THE LORENTZ FUNCTION SPACES

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ABSTRACT

Let T be an (into linear) isometry on a (real or complex) Lorentz function space $L_{w,p}$, $1 \le p < \infty$. We show that if f and g have disjoint support, then Tf and Tg also have disjoint support. Using this result, we give a characterization of the isometries of $L_{w,p}$.

1. Introduction

For any measurable function f on $(0,\infty)$, the **distribution function** d_f, and the **decreasing rearrangement** f^* of f are defined by

$$d_f(t) = \mu(|f| > t),$$
$$f^*(t) = \inf\{s > 0 \colon d_f(s) \le t\}$$

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(where μ denotes the Lebesgue measure). Let w be a strictly decreasing positive function on $(0,\infty)$ such that

$$\lim_{t\to 0} w(t)=\infty,\ \lim_{t\to\infty} w(t)=0,\ \int_0^1 w(t)dt=1,\ \text{and}\ \int_0^\infty w(t)dt=\infty.$$

For any $1\le p<\infty$, the **Lorentz space** $L_{w,p}$ is the space of all measurable functions f on $(0,\infty)$ for which

$$\|f\|=\left(\int_0^\infty (f^*)^p(t)\,w(t)dt\right)^{1/p}.$$

If $w(t)=\frac{pt^{p/q-1}}{q}$ for some $1\le p<q<\infty$, we shall denote the Lorentz space $L_{w,p}$ by $L_{q,p}$. It is known that the following mappings are isometries on real $L_{q,p}$:

(i) for any $\lambda>0$, $D_\lambda(f)(t)=\lambda^{-1/q}f(t/\lambda)$;

(ii) for any ±1-valued measurable function $\epsilon(t)$, $S_\epsilon(f)(t)=\epsilon(t)f(t)$;

(iii) for any measure-preserving transformation σ, $R_\sigma(f)(t)=f(\sigma(t))$ (for definition of measure-preserving transformation, see section 2).

Let T be an isometry on $L_{w,p}$, $1\le p<\infty$. One may ask the following question.

Question 1: Do there exist s and λ such that $(Tf)^*(t)=sf^*(t/\lambda)$ for all $f\in L_{w,p}$ and $t\in(0,\infty)$?

For any measurable function f, $\operatorname{supp} f$ denotes the set

$$\{t\colon f(t)\neq 0\}.$$

In [C-Tr], B. Turett and the first author studied the extreme points of the unit ball of $L_{q,1}$, $1<q<\infty$. They proved that T is a linear isometry from $L_{q,1}$ into itself if and only if there exists a $\lambda=\mu(\operatorname{supp}(T1_{[0,1]}))$ such that

$$(Tf)^*(t)=\lambda^{-1/q}f^*(t/\lambda).$$

Recently, S. J. Dilworth, D. A. Trautman, and the first author [C-D-T] studied the extreme points of the unit ball of $L_{w,1}[0,1]$ and they proved that if T is a surjective isometry from $L_{w,1}$ onto itself, then $(Tf)^*(t)=f^*(t)$ for all $0\le t\le 1$.

Let X be a Banach space and let x be any element on the unit sphere of X. X is said to have a **Gateaux differentiable norm** at x if for every $y\in X$

$$\lim_{\epsilon\to 0}\frac{\|x+\epsilon y\|-\|x\|}{\epsilon}$$

exists. Let $1 < p < \infty$ and let f be any element on the unit sphere of $L_{w,p}$. It is known that $L_{w,p}$ has a Gateaux differentiable norm at f if and only if $\mu\{|f| = s\} = 0$ for any $s > 0$. In this article, we study the difference

$$\lim_{\epsilon\downarrow 0} \frac{\|f+\epsilon g\|^p - \|f\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0} \frac{\|f+\epsilon g\|^p - \|f\|^p}{\epsilon}$$

for some f, $g \in L_{w,p}$ (particularly, f and g are characteristic functions) and we prove the following theorem.

MAIN THEOREM: *Let T be a linear isometry from $L_{w,p}$, $1 \le p < \infty$, into itself. Then*

$$(Tf)^*(t) = sf^*(t/\lambda)$$

where $\lambda = \mu(supp(T\,1_{(0,1)}))$ and $s = \frac{\|1_{(0,1)}\|}{\|1_{(0,\lambda)}\|}$. Moreover, if $s \neq 1$, then $w(\gamma) = s^p \lambda w(\lambda\gamma)$ almost everywhere.

This shows the answer of Question 1 is affirmative.

The remainder of the article is divided into four sections. In section 2, we study the directional derivative of $L_{w,p}$ norm, and we prove that for any f, $g \in L_{w,p}$,

$$\lim_{\epsilon\downarrow 0} \frac{\|f+\epsilon g\|^p - \|f\|^p}{\epsilon} = \lim_{\epsilon\uparrow 0} \frac{\|f+\epsilon g\|^p - \|f\|^p}{\epsilon}$$

if and only if for any $s > 0$, either $\mu(|f| = s) = 0$ or $g\,\mathrm{sgn} f$ is constant on $|f|^{-1}(s)$. In sections 3, 4 and 5, we use this result to prove the Main Theorem when $L_{w,p}$ is real $L_{w,p}(0,\infty)$, real $L_{w,p}(0,1)$, or complex $L_{w,p}$.

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2. Some lemmas

Recall a measurable mapping σ from $(0,\infty)$ to $(0,\infty)$ is said to be **measure-preserving** if for every measurable set A, $\mu(A) = \mu(\sigma^{-1}(A))$. It is known that for any measure-preserving transformation σ and any measurable function f, $f^* = (f \circ \sigma)^*$. For any two measurable functions f and g , let σ be defined by

(∗)

$$\begin{aligned}\sigma(t) =&\mu(\{t'\colon |f(t')| > |f(t)|\})\\ &+ \mu(\{t'\colon |f(t')| = |f(t)| \text{ and } g(t')\,\mathrm{sgn}(f(t')) > g(t)\,\mathrm{sgn}(f(t))\})\\ &+ \mu(\{t'\colon |f(t')| = |f(t)|,\ g(t')\,\mathrm{sgn}(f(t')) = g(t)\,\mathrm{sgn}(f(t)),\ \text{and } t' \le t\})\end{aligned}$$

where $t \in (0, \infty)$. It is easy to see that σ satisfies the following conditions:

(o) if $\mu(\operatorname{supp} f) < \infty$, then σ is a measure-preserving mapping from $\mathbb{R}$ into $\mathbb{R}$; otherwise, it is a measure-preserving mapping from $\operatorname{supp} f$ into $\mathbb{R}$;

(i) $|f| = f^* \circ \sigma$, and $\|f\| = \left(\int |f|^p w \circ \sigma\, dt\right)^{1/p}$;

(ii) $w \circ \sigma(t_1) \geq w \circ \sigma(t_2)$ if one of the following conditions holds.

(a) $|f(t_1)| > |f(t_2)|$;

(b) $|f(t_1)| = |f(t_2)|$ and $g(t_1)\operatorname{sgn}(f(t_1)) > g(t_2)\operatorname{sgn}(f(t_2))$;

(c) $|f(t_1)| = |f(t_2)|$, $g(t_1)\operatorname{sgn}(f(t_1)) = g(t_2)\operatorname{sgn}(f(t_2))$, and $t_1 \leq t_2$.

First, we assume that $\infty > p > 1$.

LEMMA 1: *Suppose that $\infty > p > 1$. Let f be a positive decreasing function in $L_{w,p}$ and let g be another function in $L_{w,p}$ such that if $s > 0$ and $\mu(f = s) \neq 0$, then g is decreasing (respectively, increasing) on $f^{-1}(s)$. Then*

$$\lim_{\epsilon \downarrow 0} \frac{\|f + \epsilon g\|^p - \|f\|^p}{\epsilon} = p \int g f^{p-1}\, w dt$$

$$\text{(respectively, } \lim_{\epsilon \uparrow 0} \frac{\|f + \epsilon g\|^p - \|f\|^p}{\epsilon} = p \int g f^{p-1}\, w dt\text{).}$$

Proof: We only prove the first equality and the other is left to the reader.

Since f is a nonnegative decreasing function,

$$\lim_{\epsilon \downarrow 0} \frac{\|f + \epsilon g\|^p - \|f\|^p}{\epsilon} \geq \lim_{\epsilon \downarrow 0} \frac{\int (|f + \epsilon g|^p - |f|^p)\, w dt}{\epsilon} = p \int g f^{p-1}\, w dt.$$

To prove the other direction, we may assume g is a bounded function. For any $\epsilon > 0$, let σ_ϵ (replace f by $f + \epsilon g$ in $(*)$) be a measure-preserving mapping which satisfies the following conditions:

(i) $|f + \epsilon g| = (f + \epsilon g)^* \circ \sigma_\epsilon$;

(ii) $w \circ \sigma_\epsilon(t_1) \geq w \circ \sigma_\epsilon(t_2)$ if one of the following conditions holds.

(a) $|(f + \epsilon g)(t_1)| > |(f + \epsilon g)(t_2)|$;

(b) $|(f + \epsilon g)(t_1)| = |(f + \epsilon g)(t_2)|$ and
$g(t_1)\operatorname{sgn}(f + \epsilon g)(t_1) > g(t_2)\operatorname{sgn}(f + \epsilon g)(t_2)$;

(c) $|(f + \epsilon g)(t_1)| = |(f + \epsilon g)(t_2)|$,
$g(t_1)\operatorname{sgn}(f + \epsilon g)(t_1) = g(t_2)\operatorname{sgn}(f + \epsilon g)(t_2)$ and $t_1 \leq t_2$.

Then there exists $0 < \alpha_\epsilon < \epsilon$ such that

$$\frac{\|f + \epsilon g\|^p - \|f\|^p}{\epsilon} \leq \frac{\int (|f + \epsilon g|^p - |f|^p)\, w \circ \sigma_\epsilon dt}{\epsilon}$$
$$= p \int g \cdot \operatorname{sgn}(f + \alpha_\epsilon g) \cdot |f + \alpha_\epsilon g|^{p-1} \cdot w \circ \sigma_\epsilon dt.$$

Since both f and w are nonnegative decreasing functions and the restriction of g to $f^{-1}(s)$ is decreasing for any $s > 0$ with $\mu(f = s) > 0$, we have $\lim_{\epsilon\downarrow 0} \sigma_\epsilon(t) = t$. The verification of

$$\lim_{\epsilon\downarrow 0} p \int g \cdot \operatorname{sgn}(f + \alpha_\epsilon g) \cdot |f + \alpha_\epsilon g|^{p-1} \cdot w \circ \sigma_\epsilon dt = p \int g f^{p-1} w dt$$

is left to the reader. ∎

Let σ be any measure-preserving transformation, and let A be any measurable set such that $A = \sigma^{-1}(B)$ for some measurable set B with $\mu(B) > 0$. A measurable function f is said to be **decreasing** (respectively, **increasing**) with respect to σ on A if for almost all t_1, $t_2 \in A$, $f(t_1) > f(t_2)$ (respectively, $f(t_1) < f(t_2)$) implies $w \circ \sigma(t_1) > w \circ \sigma(t_2)$. If $A = (0, \infty)$, then we say f is decreasing with respect to σ. Lemma 1 can be restated as:

LEMMA 2: *Let f, g be any two elements of $L_{w,p}$ and let σ, σ' be two measure preserving transformations whch satisfy the following conditions:*

(a) *$|f|$ is decreasing with respect to σ (respectively, σ');*

(b) *if $s > 0$ and $\mu(|f| = s) > 0$, then $g \operatorname{sgn} f$ is decreasing (respectively, increasing) with respect to σ (respectively, σ') on $|f|^{-1}(s)$.*

Then

$$\lim_{\epsilon\downarrow 0} \frac{\|f + \epsilon g\|^p - \|f\|^p}{\epsilon} = p \int g\, |f|^{p-1} (\operatorname{sgn} f)\, w \circ \sigma dt$$

$$(\text{respectively, } \lim_{\epsilon\uparrow 0} \frac{\|f + \epsilon g\|^p - \|f\|^p}{\epsilon} = p \int g\, |f|^{p-1} (\operatorname{sgn} f)\, w \circ \sigma' dt).$$

Remark 1: Suppose that $\infty > p > 1$. It is known that there is a measurable function f such that for any measure-preserving transformation σ, $f^* \neq |f| \circ \sigma$. But for any f, $g \in L_{w,p}$, there exist measure-preserving transformations σ and σ' which satisfy the assumptions of Lemma 2. Hence, for simplifying our proofs, we assume that for any measurable function f, there is a measure-preserving mapping σ such that $f^* = |f| \circ \sigma$. We leave all details to the reader.

Remark 2: It is known [P] that if X is a Banach space, then for any $x, y \in X$, $\|x\| = 1 = \|y\|$,

$$\lim_{\epsilon\downarrow 0} \frac{\|x + \epsilon y\|^p - \|x\|^p}{\epsilon} = p \sup\{\langle x^*, y\rangle : \|x^*\| = 1 = \langle x^*, x\rangle\}.$$

Lemma 1 is a corollary of the above well-known fact. ∎

Since w is strictly decreasing, we have the following corollaries.

COROLLARY 3: *Let f and g be any elements in $L_{w,p}$. Then*

$$\lim_{\epsilon\downarrow 0}\frac{\|f+\epsilon g\|^p-\|f\|^p}{\epsilon}=\lim_{\epsilon\uparrow 0}\frac{\|f+\epsilon g\|^p-\|f\|^p}{\epsilon}$$

if and only if for all $s>0$, $\mu(|f|=s)>0$ implies $g\cdot\operatorname{sgn} f$ is constant on $|f|^{-1}(s)$.

COROLLARY 4: *Let f be a nonnegative decreasing function in $L_{w,p}$ such that f is strictly decreasing on $(0,1)$ and $(2,\infty)$, and $f(t)=1$ if $1\le t\le 2$. For any element $g\in L_{w,p}$, let $\mathcal{B}$ be the collection of measurable functions h which satisfy the following conditions:*

(i) *$h(t)=g(t)$ if $t\in(0,1)\cup(2,\infty)$;*

(ii) *for any $r\in\mathbb{R}$, $\mu(\{t\in(1,2)\colon h(t)>r\})=\mu(\{t\in(1,2)\colon g(t)>r\})$.*

Then

$$\lim_{\epsilon\downarrow 0}\frac{\|f+\epsilon g\|^p-\|f\|^p}{\epsilon}-\lim_{\epsilon\uparrow 0}\frac{\|f+\epsilon g\|^p-\|f\|^p}{\epsilon}$$
$$=\sup_{h\in\mathcal{B}} p\int_1^2 h(t)\,w(t)dt-\inf_{h\in\mathcal{B}} p\int_1^2 h(t)\,w(t)dt.$$

Let h be any element in $\mathcal{B}$. Then

$$\lim_{\epsilon\downarrow 0}\frac{\|f+\epsilon g\|^p-\|f\|^p}{\epsilon}-\lim_{\epsilon\uparrow 0}\frac{\|f+\epsilon g\|^p-\|f\|^p}{\epsilon}$$
$$=p\int_1^2 h(t)\,w(t)dt-p\int_1^2 h(2-t)\,w(t)dt$$

if and only if h is decreasing on $(1,2)$.

COROLLARY 5: *Let A be a finite measurable subset of $(0,\infty)$, and let B be any measurable subset of A.*

$$\lim_{\epsilon\downarrow 0}\frac{\|1_A+\epsilon 1_B\|^p-\|1_A\|^p}{\epsilon}-\lim_{\epsilon\uparrow 0}\frac{\|1_A+\epsilon 1_B\|^p-\|1_A\|^p}{\epsilon}$$
$$=\sup_{C\subseteq A}\left(\lim_{\epsilon\downarrow 0}\frac{\|1_A+\epsilon 1_C\|^p-\|1_A\|^p}{\epsilon}-\lim_{\epsilon\uparrow 0}\frac{\|1_A+\epsilon 1_C\|^p-\|1_A\|^p}{\epsilon}\right)$$

if and only if $\mu(B)=\frac{1}{2}\mu(A)$.

COROLLARY 6: *Let T be an isometry on $L_{w,p}$, and let A, B be any two disjoint finite measurable sets. Let $f=T\,1_A$ and let $g=T\,1_B$. Then for any real number α, the following are equivalent:*

(a) *there exists $s>0$ such that $\mu(|f+\alpha g|=s)>0$ and $g\operatorname{sgn}(f+\alpha g)$ is not constant on $|f+\alpha g|^{-1}(s)$;*

(b) $\lim_{\epsilon\downarrow 0} \frac{\|(f+\alpha g)+\epsilon g\|^p - \|f+\alpha g\|^p}{\epsilon} \neq \lim_{\epsilon\uparrow 0} \frac{\|(f+\alpha g)+\epsilon g\|^p - \|f+\alpha g\|^p}{\epsilon}$;

(c) $\lim_{\epsilon\downarrow 0} \frac{\|(1_A+\alpha 1_B)+\epsilon 1_B\|^p - \|1_A+\alpha 1_B\|^p}{\epsilon} \neq \lim_{\epsilon\uparrow 0} \frac{\|(1_A+\alpha 1_B)+\epsilon 1_B\|^p - \|1_A+\alpha 1_B\|^p}{\epsilon}$;

(d) $\alpha = \pm 1$.

Remark 3: Let s be any positive real number such that $\mu(|f| = s) > 0$. By Corollary 6, $g \cdot \operatorname{sgn} f|_{|f|^{-1}(s)}$ is constant. If $g \cdot \operatorname{sgn} f|_{|f|^{-1}(s)} \neq 0$, then

(i) $\operatorname{sgn} g|_{|f|^{-1}(s)} = \pm \operatorname{sgn} f|_{|f|^{-1}(s)}$;

(ii) $|g|$ is constant on $|f|^{-1}(s)$. ∎

If $p = 1$, Lemma 1 becomes

LEMMA 7: *Let f be a positive decreasing function in $L_{w,1}$ and let g be any function in $L_{w,1}$ such that*

(i) *if $s \neq 0$ and $\mu(f = s) > 0$, then $g\ \big|_{f^{-1}(s)}$ is decreasing (respectively, increasing);*

(ii) *$|g|\ \big|_{f^{-1}(0)}$ is decreasing.*

Then

$$\lim_{\epsilon\downarrow 0} \frac{\|f+\epsilon g\| - \|f\|}{\epsilon} = \int_{\operatorname{supp} f} g\, w\, dt + \int_{\mathbb{R}\setminus \operatorname{supp} f} |g|\, w\, dt$$

$$(\textit{respectively, } \lim_{\epsilon\uparrow 0} \frac{\|f+\epsilon g\| - \|f\|}{\epsilon} = \int_{\operatorname{supp} f} g\, w\, dt + \int_{\mathbb{R}\setminus \operatorname{supp} f} |g|\, w\, dt).$$

By Lemma 7, one easily sees that Corollaries 3–6 hold if $p = 1$. We leave all details to reader.

3. The isometries on $L_{w,p}(0,\infty)$, $1 \leq p < \infty$

Let T be any isometry on $L_{w,p}(0,\infty)$. It is known that $L_{w,p}$ is separable. To prove Main Theorem, it is enough to show that

(i) there is $s > 0$ such that for any finite measurable set A, $|T 1_A| = s\, 1_B$ where $B = \operatorname{supp}(T 1_A)$;

(ii) if A, B are two disjoint finite measurable sets, then $T 1_A$ and $T 1_B$ are disjoint.

We do not know any easy way to prove (i) and (ii). Our proof is divided into seven steps. In the first three steps, we show that there is a unique $s > 0$ such that $\mu(|T 1_A| = s) > 0$ for any finite measurable set A. Moreover, if A and B

are two disjoint finite measurable subsets of $(0,\infty)$, then the restriction of $T\,1_B$ to $|T\,1_A|^{-1}(s)$ is zero. (This is equivalent to $\{t\colon |T\,1_A|(t)=s\}\cup\{t\colon |T\,1_B|(t)=s\}\subseteq\{t\colon |T\,1_{A\cup B}|(t)=s\}$.) Next, we show that if A and B are two disjoint finite measurable sets, then $|T\,1_{A\cup B}|^{-1}(s)=|T\,1_A|^{-1}(s)\cup|T\,1_B|^{-1}(s)$. The proof is quite long, and we do not know any simple proof. The proof contains two steps. Let A be any finite measurable set and let A_1, A_2, A_3 be three disjoint proper measurable subsets of A such that $A=A_1\cup A_2\cup A_3$. In step 4, we show that for any $t\in|T\,1_A|^{-1}(s)$, one of $T\,1_{A_1}(t)$, $T\,1_{A_2}(t)$, $T\,1_{A_3}(t)$ must be zero. (This is equivalent to at most two of $T\,1_{A_1}(t)$, $T\,1_{A_2}(t)$, $T\,1_{A_3}(t)$ are nonzero.) In step 5, we show that at most one (so exact one) of them is nonzero. The remainder two steps show that $\mu(\{t\colon 0<|T\,1_A|(t)\neq s\})=0$. Let A be any finite measurable subset of $(0,\infty)$. In step 6, we show that for any measurable subset B of A,

$$\frac{\mu(|T\,1_A|=s)}{\mu(A)}=\frac{\mu(|T\,1_B|=s)}{\mu(B)}.$$

(Note: this implies that $\frac{\mu(|T\,1_A|=s)}{\mu(A)}=\frac{\mu(|T\,1_B|=s)}{\mu(B)}$ for any two finite measurable subsets A, B of $(0,\infty)$.) Using this result, we show (step 7) that $w(\gamma)=s^p\lambda w(\lambda\gamma)$. This implies for any finite measurable set A,

$$\|1_A\|=\|s1_{|T\,1_A|^{-1}(s)}\|\le\|T\,1_A\|=\|1_A\|.$$

So we must have $|T\,1_A|=s\,1_{|T\,1_A|^{-1}(s)}$.

(1) We claim that for any finite measurable set A, there exists $s>0$ such that $\mu(|T\,1_A|=s)>0$.

Let A_1, A_2 be two disjoint subsets of A such that $A_1\cup A_2=A$. We denote $T\,1_{A_1}$, $T\,1_{A_2}$ and $T\,1_A$ by f_1, f_2 and f. By Corollary 6, there exists $s>0$ such that $\mu(|f|=s)>0$, and the restriction of $f_1\operatorname{sgn} f$ to $|f|^{-1}(s)$ is not constant. We proved our claim.

(2) Let A, A_1, f, f_1 and s be given as above. Let B be another finite measurable subset of $(0,\infty)$ such that $B\cap A=\emptyset$, and let $g=T\,1_B$. We claim that $g\,\big|_{|f|^{-1}(s)}=0$.

Suppose that it is not true. By Remark 3, $\operatorname{sgn} g\,\big|_{|f|^{-1}(s)}=\pm\operatorname{sgn} f\,\big|_{|f|^{-1}(s)}$ and the restriction of $|g|$ to the set $|f|^{-1}(s)$ is constant. So there exists $r>0$ such that $|f|^{-1}(s)\subseteq|g|^{-1}(r)$. Note: A_1 and B are disjoint. By Remark 3 (or Corollary 6), the restriction of $f_1\operatorname{sgn} g$ to $|g|^{-1}(r)$ is constant. This implies

$$f_1\cdot\operatorname{sgn} f\,\big|_{|f|^{-1}(s)}=\pm f_1\cdot\operatorname{sgn} g\,\big|_{|f|^{-1}(s)}$$

is constant. We get a contradiction. So $g \,\big|_{|f|^{-1}(s)} = 0$.

Note: we did not prove that for any $s > 0$, $\mu(|T\,1_A| = s) > 0$ implies $T\,1_B = 0$ on $|T\,1_A|^{-1}(s)$ for every B such that $A \cap B = \emptyset$. We only proved that it is true for some $s > 0$.

(3) We claim that there is a unique $s > 0$ such that for every finite measurable set A

(i) if $\mu(A) > 0$, then $\mu(|T\,1_A| = s) > 0$;
(ii) if $r > 0$ and $r \neq s$, then $\mu(|T\,1_A| = r) = 0$;
(iii) for any two disjoint measurable sets A and B, $|T\,1_A|^{-1}(s)$ and $|T\,1_B|^{-1}(s)$ are disjoint.

Let s be the number given as above. Suppose $r \neq 0$ and $\mu(|g|^{-1}(r)) > 0$. By (2), $g|_{|f|^{-1}(s)} = 0$. Since B and A are disjoint, by Corollary 6, there exist $c \geq 0$ and $a = \pm 1$ such that

$$f \,\big|_{|g|^{-1}(r)} = a\,c \cdot \operatorname{sgn} g \, 1_{|g|^{-1}(r)}.$$

Hence, the restrictions of $|f + \frac{a\,(s-c)}{r}\, g|$ and $|f + \frac{a\,(-s-c)}{r}\, g|$ to the set $|f|^{-1}(s) \cup |g|^{-1}(r)$ are constant. By Corollary 6 again, we have

$$\frac{s-c}{r} = \pm 1 \quad \text{and} \quad \frac{-s-c}{r} = -1.$$

But $s \neq 0$. So $c = 0$ and $r = s$. We proved that for a fixed finite measurable set A, there is $s > 0$ such that for any $r > 0$ and any finite measurable set B which is disjoint with A,

$$\mu(|T\,1_B| = r) > 0 \quad \text{if and only if } r = s.$$

The remainder of the proof is left to the reader.

From now on, s is a fixed positive number such that $\mu(|T\,1_A| = s) > 0$ for every finite measurable set A, $\mu(A) > 0$.

(4) Let A_1, A_2, A_3 be three disjoint proper subsets of A such that $A = A_1 \cup A_2 \cup A_3$. We denote $T\,1_{A_1}$, $T\,1_{A_2}$ and $T\,1_{A_3}$ by f_1, f_2 and f_3. Without loss of generality, we assume that f is a nonnegative decreasing function and the restriction of $f_1 - f_2$ to the set $f^{-1}(s) = [a, b]$ is decreasing.

We claim that

(a) both f_1 and $-f_2$ are decreasing on $f^{-1}(s)$;
(b) $f_1 f_2 f_3 \,\big|_{[a,b]} = 0$. (So for every $t \in [a, b]$, one of $f_1(t)$, $f_2(t)$, $f_3(t)$ must be zero.)

Suppose claim (a) were proved. If A_1, A_2 are two disjoint measurable subsets of A, and if $f = T1_A$, $f_1 = T1_{A_1}$, $f_2 = T1_{A_2}$, then there exists a measure-preserving transformation σ such that

(i) $|f|$ is decreasing with respect to σ;

(ii) $(f_1 - f_2)\operatorname{sgn} f$, $f_1 \operatorname{sgn} f$ and $-f_2 \operatorname{sgn} f$ are decreasing with respect to σ on $|f|^{-1}(s)$.

Proof of (a): Without loss of generality, we may assume that $A_1 = (0, c]$, $A_3 = (c, d)$ and $A_2 = [d, 1]$. Then

$$
\begin{aligned}
&p\int_a^b f_1(t)f^{p-1}(t)w(t)dt - p\int_a^b f_1(a+b-t)f^{p-1}(t)w(t)dt\\
&\quad + p\int_a^b (-f_2)(t)f^{p-1}(t)w(t)dt - p\int_a^b (-f_2)(a+b-t)f^{p-1}(t)w(t)dt\\
&= p\int_a^b (f_1 - f_2)(t)f^{p-1}(t)w(t)dt - p\int_a^b (f_1 - f_2)(a+b-t)f^{p-1}(t)w(t)dt\\
&= \lim_{\epsilon\downarrow 0}\frac{\|f + \epsilon(f_1 - f_2)\|^p - \|f\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0}\frac{\|f + \epsilon(f_1 - f_2)\|^p - \|f\|^p}{\epsilon}\\
&= \lim_{\epsilon\downarrow 0}\frac{\|1_A + \epsilon(1_{A_1} - 1_{A_2})\|^p - \|1_A\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0}\frac{\|1_A + \epsilon(1_{A_1} - 1_{A_2})\|^p - \|1_A\|^p}{\epsilon}\\
&= p\int_0^1 (1_{A_1} - 1_{A_2})(t)w(t)dt - p\int_0^1 (1_{A_1} - 1_{A_2})(1-t)w(t)dt\\
&= p\int_0^1 1_{A_1}(t)w(t)dt - p\int_0^1 1_{A_1}(1-t)w(t)dt\\
&\qquad\qquad + p\int_0^1 (-1_{A_2})(t)w(t)dt - p\int_0^1 (-1_{A_2})(1-t)w(t)dt\\
&= \lim_{\epsilon\downarrow 0}\frac{\|1_A + \epsilon 1_{A_1}\|^p - \|1_A\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0}\frac{\|1_A + \epsilon 1_{A_1}\|^p - \|1_A\|^p}{\epsilon}\\
&\quad + \lim_{\epsilon\downarrow 0}\frac{\|1_A - \epsilon 1_{A_2}\|^p - \|1_A\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0}\frac{\|1_A - \epsilon 1_{A_2}\|^p - \|1_A\|^p}{\epsilon}\\
&= \lim_{\epsilon\downarrow 0}\frac{\|f + \epsilon f_1\|^p - \|f\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0}\frac{\|f + \epsilon f_1\|^p - \|f\|^p}{\epsilon}\\
&\quad + \lim_{\epsilon\downarrow 0}\frac{\|f - \epsilon f_2\|^p - \|f\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0}\frac{\|f - \epsilon f_2\|^p - \|f\|^p}{\epsilon}.
\end{aligned}
$$

By Corollary 4, we have

$$
\begin{aligned}
&p\int_a^b f_1(t)f^{p-1}(t)w(t)dt - p\int_a^b f_1(a+b-t)f^{p-1}(t)w(t)dt\\
&= \lim_{\epsilon\downarrow 0}\frac{\|f + \epsilon f_1\|^p - \|f\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0}\frac{\|f + \epsilon f_1\|^p - \|f\|^p}{\epsilon},
\end{aligned}
$$

and

$$p\int_a^b (-f_2)(t)f^{p-1}(t)w(t)dt - p\int_a^b (-f_2)(a+b-t)f^{p-1}(t)w(t)dt = \lim_{\epsilon\downarrow 0}\frac{\|f-\epsilon f_2\|^p - \|f\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0}\frac{\|f-\epsilon f_2\|^p - \|f\|^p}{\epsilon}.$$

By Corollary 4 again, both f_1 and $-f_2$ are decreasing on $[a,b]$. We proved (a).

Proof of (b): Note that

(iii) f_1 and $-f_2$ are decreasing on $[a,b]$;
(iv) $|f_3|^{-1}(s) \subseteq [a,b]$;
(v) since $\mu(|f_3| = s) > 0$, both f_1 and f_2 are zero on $|f_3|^{-1}(s)$.

There exist c', d', $a \le c' < d' \le b$, such that f_1 and f_2 are zero on (c', d'). So $-f_1 f_2$ must be nonnegative on $[a,b]$. Similarly, $-f_3 f_2$ and $-f_3 f_1$ are nonnegative on $[a,b]$ (since any rearrangement does not change the sign). This implies $-(f_1 f_2 f_3)^2$ is nonnegative on $[a,b]$. And $f_1 f_2 f_3$ must be zero on $[a,b]$. We proved our claim.

(5) We claim that for any $t \in [a,b]$, exact one of $f_1(t)$, $f_2(t)$, $f_3(t)$ is nonzero. Suppose that the claim were proved. Then for any two disjoint finite measurable sets A and B, we have

(i) $|T1_A|^{-1}(s)$ and $|T1_B|^{-1}(s)$ are disjoint;
(ii) $|T1_{A\cup B}|^{-1}(s) = |T1_A|^{-1}(s) \cup |T1_B|^{-1}(s)$;
(iii) $T1_A \cdot \mathrm{sgn}(T1_{A\cup B})\big|_{|T1_{A\cup B}|^{-1}(s)} = s\,1_{|T1_A|^{-1}(s)}$.

Suppose the claim is not true. Without loss of generality, we may assume that there exists t, $a < t < b$, such that

$$f_1(t) + f_2(t) = s, \quad f_1(t) > 0 \quad \text{and} \quad f_2(t) < 0.$$

Since both f_1 and $-f_2$ are decreasing on $[a,b]$, for any $t' \in [a,t)$,

$$f_1(t') > s, \quad f_2(t') < 0, \quad \text{and} \quad f_3(t') = 0.$$

Let $[a,c) = \{t' \in [a,b]\colon f_1(t') > s\}$. (So $c > a$.) Then

(iv) f_1 is strictly decreasing and $f_1 > s$ on $[a,c)$.

SUBCLAIM 1: $f_3|_{[a,c)} = 0$. Note: f_3 is zero on $[a, t]$. Suppose Subclaim 1 is not true. Then there exists $r \in (t, c)$ such that $f_3(r) < 0$ and $f_2(r) = 0$. Note: the $-f_2$ is decreasing on $[a, b]$, and $f_1 + f_2 + f_3 = s$ on $[a, b]$. So we have

(v) f_3 is negative on (r, c) (because $f_2 \geq 0$ and $f_1 > s$ on (r, c)).

(Note again: f_3 is zero on $[a, t)$ and $t < r$.) So for any measure-preserving mapping σ on $[a, b]$, either $f_1 \circ \sigma$ is not decreasing or $(-f_3) \circ \sigma$ is not decreasing. We get a contradiction.

Let A_4 and A_5 be two disjoint measurable subsets of A_2 such that $A_2 = A_4 \cup A_5$.

SUBCLAIM 2: If $T 1_{A_4}|_{[a,c)} \neq 0$, then $T 1_{A_5}|_{[a,c)} = 0$. By (4), there is a measure-preserving transformation σ on $[a, b]$ such that both $(T1_{A_1}) \circ \sigma$ and $(-T1_{A_4}) \circ \sigma$ are decreasing. Since f_1 is strictly decreasing on $[a, c]$ the restriction of σ to $[a, c]$ is identity. But $T 1_{A_4}|_{[a,c)} \neq 0$. So the above proof shows $T 1_{A_5 \cup A_3}|_{[a,c)} = 0$ and $T 1_{A_5}|_{[a,c)} = 0$ (note: we already proved that $T 1_{A_3}$ is zero on $[a, c]$). We proved subclaim 2.

Since A has finite measure, for any $\epsilon > 0$, there is a partition $\{B_1, B_2, \ldots, B_k\}$ of A_2 such that $\mu(B_j) < \epsilon$ for all $1 \leq j \leq k$. The above argument shows that for some $1 \leq j \leq k$, $T 1_{B_j}|_{[a,c]} = f_2|_{[a,c]}$. But T is an isometry. This is impossible. We proved our claim.

(6) Let A be any finite measurable set and let $f = T 1_A$. We claim that for any measurable subset B of A,

$$\frac{\mu(|T 1_A| = s)}{\mu(A)} = \frac{\mu(|T 1_B| = s)}{\mu(B)}.$$

(5) shows that for any subset B of A

$$T 1_B \operatorname{sgn} f \,\big|_{|f|^{-1}(s)} = s\, 1_{|T 1_B|^{-1}(s)}.$$

Since T is continuous, there exists a measurable subset A_1 of A such that $\mu(|T 1_{A_1}| = s) = \frac{1}{2}\mu(|f| = s)$. So

$$\begin{aligned}
&\lim_{\epsilon \downarrow 0} \frac{\|1_A + \epsilon 1_{A_1}\|^p - \|1_A\|^p}{\epsilon} - \lim_{\epsilon \uparrow 0} \frac{\|1_A + \epsilon 1_{A_1}\|^p - \|1_A\|^p}{\epsilon} \\
&= \lim_{\epsilon \downarrow 0} \frac{\|f + \epsilon T 1_{A_1}\|^p - \|f\|^p}{\epsilon} - \lim_{\epsilon \uparrow 0} \frac{\|f + \epsilon T 1_{A_1}\|^p - \|f\|^p}{\epsilon} \\
&= \sup_{B \subseteq A} \left(\lim_{\epsilon \downarrow 0} \frac{\|f + \epsilon T 1_B\|^p - \|f\|^p}{\epsilon} - \lim_{\epsilon \uparrow 0} \frac{\|f + \epsilon T 1_B\|^p - \|f\|^p}{\epsilon} \right) \\
&= \sup_{B \subseteq A} \left(\lim_{\epsilon \downarrow 0} \frac{\|1_A + \epsilon 1_B\|^p - \|1_A\|^p}{\epsilon} - \lim_{\epsilon \uparrow 0} \frac{\|1_A + \epsilon 1_B\|^p - \|1_A\|^p}{\epsilon} \right).
\end{aligned}$$

By Corollary 5, $\mu(A_1) = \frac{1}{2}\mu(A)$. Hence, for any subset A_1 of A, $\mu(A_1) = \frac{1}{2}\mu(A)$ if and only if $\mu(|T 1_{A_1}| = s) = \frac{1}{2}\mu(|T 1_A| = s)$. By induction, we have

$$\mu(A_1) = \frac{1}{2^n}\mu(A) \quad \text{if and only if } \mu(|T 1_{A_1}| = s) = \frac{1}{2^n}\mu(|T 1_A| = s).$$

Since T is continuous, we must have

$$\frac{\mu(|T 1_A| = s)}{\mu(A)} = \frac{\mu(|T 1_B| = s)}{\mu(B)}$$

for any measurable subset B of A. Let $\lambda = \frac{\mu(|T 1_A|=s)}{\mu(A)}$. The proof also shows for any finite measurable subset B of $(0, \infty)$,

$$\frac{\mu(|T 1_B| = s)}{\mu(B)} = \lambda.$$

(7) We claim that for any $\gamma > 0$, $w(\gamma) = s^p \lambda w(\lambda\gamma)$. Let A be any finite measurable set and let $\beta = \mu(|T(1_A)| > s)$. If A_1 is a measurable subset of A, by (5) and (6)

(i) $\{t\colon |T 1_{A_1}|(t) = s\} \subseteq \{t\colon |T 1_A|(t) = s\}$;

(ii) $\mu(|T 1_A| = s) = \lambda\mu(A)$, and $\mu(|T 1_{A_1}| = s) = \lambda\mu(A_1)$.

(iii) $T 1_A \cdot \operatorname{sgn}(T 1_{A\cup B}) \big|_{|T 1_{A\cup B}|^{-1}(s)} = s\, 1_{|T 1_A|^{-1}(s)}$.

Hence, if $\mu(A_1) = \gamma \leq \frac{\mu(A)}{2}$, then

$$(**)\quad \begin{aligned} & p\int_0^{\gamma} w(t)dt - p\int_{\mu(A)-\gamma}^{\mu(A)} w(t)dt \\ &= \lim_{\epsilon\downarrow 0} \frac{\|1_A + \epsilon 1_{A_1}\|^p - \|1_A\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0} \frac{\|1_A + \epsilon 1_{A_1}\|^p - \|1_A\|^p}{\epsilon} \\ &= \lim_{\epsilon\downarrow 0} \frac{\|T 1_A + \epsilon T 1_{A_1}\|^p - \|T 1_A\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0} \frac{\|T 1_A + \epsilon T 1_{A_1}\|^p - \|T 1_A\|^p}{\epsilon} \\ &= p\int_{\beta}^{\lambda\gamma+\beta} s^p w(t)dt - p\int_{\beta+\lambda(\mu(A)-\gamma)}^{\beta+\lambda\mu(A)} s^p w(t)dt. \end{aligned}$$

Differentiating both sides with respect to γ, we get

$$(***)\qquad w(\gamma) - w(\mu(A) - \gamma) = s^p \lambda[w(\beta + \lambda\gamma) - w(\beta + \lambda(\mu(A) - \gamma))].$$

Let $\{A_i\}$ be a sequence of finite measurable subsets of $(0, \infty)$ such that $\lim_{i\to\infty} \mu(A_i) = \infty$, and let $\beta_i = \mu(|T 1_{A_i})| > s)$. Then for any $\gamma > 0$,

$$w(\gamma) - w(\mu(A_i) - \gamma) \leq s^p \lambda w(\beta_i + \lambda\gamma).$$

Since $\lim_{t\to\infty} w(t) = 0$, $\{\beta_i: i \in \mathbb{N}\}$ is bounded. Without loss of generality, we may assume that $\{\beta_i\}$ converges to β. Then

$$\begin{aligned} w(\gamma) &= \lim_{i\to\infty} (w(\gamma) - w(\mu(A_i) - \gamma)) \\ &= \lim_{i\to\infty} s^p \lambda [w(\beta_i + \lambda\gamma) - w(\beta_i + \lambda(\mu(A_i) - \gamma))] \\ &= s^p \lambda w(\beta + \lambda\gamma) \end{aligned}$$

for almost all $\gamma \in \mathbb{R}$. So for any finite measurable subset A of $(0, \infty)$,

$$\begin{aligned} \|1_A\|^p &= \int_0^{\mu(A)} w(t)dt \\ &= \int_0^{\mu(A)} s^p \lambda w(\beta + \lambda t)dt \\ &= \int_0^{\lambda\mu(A)} s^p w(\beta + t)dt \\ &= \int_\beta^{\beta+\lambda\mu(A)} s^p w(t)dt \\ &\le \int_0^{\lambda\mu(A)} s^p w(t)dt \\ &= \|s\, 1_{|T1_A|^{-1}(s)}\|^p \le \|T1_A\|^p. \end{aligned}$$

But T is an isometry. So we must have

$$\beta = 0, \quad \|s\, 1_{(0,\lambda\mu(A))}\| = \|T1_A\|, \quad \text{and} \quad (T1_A)^* = s\, 1_{(0,\lambda\mu(A))}.$$

We proved our Main Theorem.

4. The isometries on $L_{w,p}(0,1)$

Let T be an isometry on $L_{w,p}(0,1)$, and let A be any finite measurable subset of $(0,1)$. In section 3, we proved that there exist $s > 0$ and $\lambda > 0$ such that if A_1 is a measurable subset of A, then

(i) $\{t\colon |T1_{A_1}|(t) = s\} \subseteq \{t\colon |T1_A|(t) = s\}$;
(ii) $\mu(|T1_A| = s) = \lambda\mu(A)$, and $\mu(|T1_{A_1}| = s) = \lambda\mu(A_1)$;
(iii) $T1_A \cdot \operatorname{sgn}(T1_{A\cup B})\,\big|_{|T1_{A\cup B}|^{-1}(s)} = s\, 1_{|T1_A|^{-1}(s)}$.

So we have $1 \ge \mu(|T1_{(0,1)}| = s) = \lambda$.

To characterize the isometries on $L_{w,p}(0,1)$, we need more work. We divide the proof into six steps. In the first 4 steps (Steps 8–11), we consider the following four special cases:

(iv) $s = 1$.
(v) T is a surjective isometry.
(vi) $L_{w,p} = L_{q,p}$.
(vii) There is $c > 0$ such that w is linear on $(0, c)$. (Note: in this article, we do not use the fact $\lim_{t\to 0^+} w(t) = \infty$.)

In Step 11, we show that if w is linear on $(0, c)$ for some $c > 0$, then $s = 1$. Hence, for any measurable subset A of $(0,1)$, $(T\,1_A)^* = 1_{(0,\mu(A))}$. (This is a fact which is proved in Step 8.)

In Step 12 and Step 13, we show that $\mu(|T\,1_A| > s) = 0$ and $\mu(0 < |T\,1_A| < s) = 0$ for every measurable subset A of $(0,1)$.

Assume the above claims were proved. Then $(T\,1_A)^* = s\,1_{(0,\lambda\,\mu(A))}$. This implies

$$\int_0^{\gamma} w(t)dt = \int_0^{\lambda\gamma} s^p w(t)dt.$$

So $w(\gamma) = s^p\,\lambda\, w(\lambda\gamma)$.

(8) We claim that if $s = 1$, then for any measurable subset A of $(0,1)$, $(T\,1_A)^* = 1_{(0,\mu(A))}$. Let β, γ be two positive numbers such that $\beta + \gamma \le 1$. It is easy to see that

(i) if $\beta + \frac{\gamma}{2} \le \frac{1}{2}$, then

$$\begin{aligned}\int_{\beta}^{\beta+\frac{\gamma}{2}} w(t)dt - \int_{\beta+\frac{\gamma}{2}}^{\beta+\gamma} w(t)dt &\le \int_{\beta}^{\beta+\frac{\gamma}{2}} w(t)dt - \int_{1-\beta-\frac{\gamma}{2}}^{1-\beta} w(t)dt \\ &\le \int_0^{\frac{1}{2}} w(t)dt - \int_{\frac{1}{2}}^{1} w(t)dt;\end{aligned}$$

(ii) if $\beta + \frac{\gamma}{2} > \frac{1}{2}$, then

$$\begin{aligned}\int_{\beta}^{\beta+\frac{\gamma}{2}} w(t)dt - \int_{\beta+\frac{\gamma}{2}}^{\beta+\gamma} w(t)dt &\le \int_{1-\beta-\gamma}^{1-\beta-\frac{\gamma}{2}} w(t)dt - \int_{\beta+\frac{\gamma}{2}}^{\beta+\gamma} w(t)dt \\ &\le \int_0^{\frac{1}{2}} w(t)dt - \int_{\frac{1}{2}}^{1} w(t)dt.\end{aligned}$$

So we have

$$\int_{\beta}^{\beta+\frac{\gamma}{2}} w(t)dt - \int_{\beta+\frac{\gamma}{2}}^{\beta+\gamma} w(t)dt \le \int_0^{\frac{1}{2}} w(t)dt - \int_{\frac{1}{2}}^{1} w(t)dt,$$

and the equality holds if and only if $\beta = 0$ and $\gamma = 1$. By (**) (with $A = (0,1)$, $A_1 = (0, \frac{1}{2})$),

$$\int_0^{1/2} w(t)dt - \int_{1/2}^{1} w(t)dt = \int_{\beta}^{\beta+\frac{\lambda}{2}} s^p w(t)dt - \int_{\beta+\frac{\lambda}{2}}^{\beta+\lambda} s^p w(t)dt.$$

So we must have $s \ge 1$. Moreover, if $s = 1$, then $\lambda = 1$. Hence, if $s = 1$, then

$$T\,1_{(0,1)} = 1_{(0,1)} \quad \text{and} \quad (T\,1_A)^* = 1_{(0,\mu(A))}$$

for every measurable subset A of $(0,1)$.

(9) Let T be a surjective isometry on $L_{w,p}(0,1)$. Since T^{-1} is also an isometry, there is $s' > 0$ such that $\mu(|T^{-1}\,1_{(0,1)}| = s') > 0$. It is known that $s' \ge 1$. We claim that $s' = s^{-1}$. If the claim is true, we must have $s = 1$. By (8), $(T\,1_A)^* = 1_{(0,\mu(A))}$ for every measurable subset A of $(0,1)$.

In section 3, we proved that for any (finite) measurable set A and any function f in $L_{w,p}$, if $\operatorname{supp} f$ is disjoint with A, then Tf is zero on $|T\,1_A|^{-1}(s)$. Let $B_1 = |T\,1_{(0,1)}|^{-1}(s)$, and let $B_2 = |T^{-1}(T\,1_{(0,1)}\cdot 1_{B_1})|^{-1}(s\,s')$. The above remark shows

$$T^{-1}\left((T\,1_{(0,1)}) \cdot 1_{(0,1)\setminus B_1}\right) = 0 \quad \text{on } B_2.$$

But $T^{-1}T\,1_{(0,1)} = 1_{(0,1)}$. So we must have $s\,s' = 1$, and we proved our claim.

(10) Let $L_{w,p}$ be an $L_{q,p}$-space. It is easy to see that

$$\begin{aligned}\int_0^{\lambda/2} \frac{p\,t^{p/q-1}}{q}dt - \int_{\lambda/2}^{\lambda} \frac{p\,t^{p/q-1}}{q}dt \\ = \sup_{0\le\beta\le\beta+\lambda\le 1}\left(\int_{\beta}^{\beta+\lambda/2} \frac{p\,t^{p/q-1}}{q}dt - \int_{\beta+\lambda/2}^{\beta+\lambda} \frac{p\,t^{p/q-1}}{q}dt\right).\end{aligned}$$

By (**) (with $A = (0,1)$, $A_1 = (0, \frac{1}{2})$)

$$\begin{aligned} 2^{1-\frac{p}{q}} - 1 &= \int_0^{1/2} \frac{p\,t^{p/q-1}}{q}dt - \int_{1/2}^{1} \frac{p\,t^{p/q-1}}{q}dt \\ &\le \int_0^{\lambda/2} \frac{p\,s^p\,t^{p/q-1}}{q}dt - \int_{\lambda/2}^{\lambda} \frac{p\,s^p\,t^{p/q-1}}{q}dt \\ &= s^p[2(\frac{\lambda}{2})^{p/q} - \lambda^{p/q}]. \end{aligned}$$

This implies $s \geq \lambda^{-1/q} \geq 1$, and

$$\|T\,1_{(0,1)}\|^p \geq \|1_{|T\,1_{(0,1)}|^{-1}(s)}\| = \int_0^\lambda \frac{ps^p t^{p/q-1}}{q} dt = s^p \lambda^{p/q} \geq 1 = \|1_{(0,1)}\|^p.$$

Since T is an isometry, we must have

$$s = \lambda^{-1/q}, \quad (T\,1_A)^* = \lambda^{-1/q}\,1_{(0,\lambda\mu(A))} \quad \text{and} \quad (T\,f)^*(t) = \lambda^{-1/q} f^*(t/\lambda).$$

(11) Suppose that w is linear on $(0,c)$ for some $c > 0$. We claim that $s = 1$. Let $w(t) = b - at$ whenever $0 < t < c$. Let A be any measurable subset of $(0,1)$ such that $\mu(A) < c$. Let $\beta = \mu(|T\,1_A| > s)$. Since $\|1_A\| = \|T\,1_A\|$ and $s \geq 1$,

$$\beta + \lambda\mu(A) = \mu(|T\,1_A| \geq s) \leq c.$$

By $(***)$, for almost all $t \in (0, \frac{\mu(A)}{2})$

$$\begin{aligned} a(\mu(A) - 2t) =& b - at - (b - a(\mu(A) - t)) \\ =& s^p\lambda[b - a(\beta + \lambda t) - (b - a(\beta + \lambda(\mu(A) - t)))] \\ =& s^p\lambda^2 a(\mu(A) - 2t). \end{aligned}$$

Since $a > 0$, we have $s^p\lambda^2 = 1$. So

$$\begin{aligned} \|T\,1_A\|^p &= \int_0^{\mu(\mathrm{supp}\,(T\,1_A))} ((T\,1_A)^*)^p(t)(b - at)dt \\ &\geq \int_0^{\mu(A)\lambda} s^p(b - at)dt \\ &= \int_0^{\mu(A)\lambda} \frac{1}{\lambda^2}(b - at)dt \\ &\geq \int_0^{\mu(A)} b - at\,dt = \|1_A\|^p, \end{aligned}$$

and the equality holds if and only if $\lambda = 1$ (and $s = 1$). We proved our claim.

Now, we prove the general case. We assume that $s > 1$.

(12) Let A be any measurable subset of $(0,1)$. We claim that $\mu(|T\,1_A| > s) = 0$.

Let A_1, A_2 and A_3 be three disjoint measurable subsets of A such that $A_1 \cup A_2 \cup A_3 = A$. First let $A_4 = A_1 \cup A_2$ and A_3 be two fixed disjoint sets. (So $A = A_4 \cup A_3$.) For $1 \geq \alpha \geq 0$, let

$$\begin{aligned} \beta_\alpha &= \mu\big(|T(1_{A_1\cup A_2} + \alpha 1_{A_3})| > s\big), \\ \gamma &= \mu(A_1). \end{aligned}$$

By Corollary 6 and (5), we have

(i) $\mu(|T(1_{A_4} + \alpha 1_{A_3})| = r) > 0$ if and only if $r = s$ or $r = \alpha s$.

(ii) $T1_{A_1}\big|_{|T1_{A_3}|^{-1}(s)} = 0$. (So $T1_{A_1}\big|_{|T(1_{A_4}+\alpha 1_{A_3})|^{-1}(\alpha s)} = 0$.)

(iii) $\mu(|T(1_{A_4} + \alpha 1_{A_3})| = s) = \lambda\mu(A_4)$ when $0 \le \alpha < 1$.

Let A_4 be a fixed measurable subset of A. We claim that

(a) $w(\gamma) - w(\mu(A_4) - \gamma) = s^p\lambda w(\beta_\alpha + \gamma\lambda) - s^p\lambda w(\beta_\alpha + (\mu(A_4) - \gamma)\lambda)$ for any γ, $0 < \gamma < \frac{\mu(A_4)}{2}$;

(b) the function $\alpha \to \beta_\alpha$ is continuous.

Proof of claim (a): By Corollary 4, for $1 > \alpha > 0$ and $\gamma < \frac{\mu(A_4)}{2}$,

$$\begin{aligned}
&p\int_0^\gamma w(t)dt - p\int_{\mu(A_4)-\gamma}^{\mu(A_4)} w(t)dt\\
&= \lim_{\epsilon\downarrow 0}\frac{\|(1_{A_4}+\alpha 1_{A_3}) + \epsilon 1_{A_1}\|^p - \|(1_{A_4}+\alpha 1_{A_3})\|^p}{\epsilon}\\
&\qquad - \lim_{\epsilon\uparrow 0}\frac{\|(1_{A_4}+\alpha 1_{A_3}) + \epsilon 1_{A_1}\|^p - \|(1_{A_4}+\alpha 1_{A_3})\|^p}{\epsilon}\\
&= \lim_{\epsilon\downarrow 0}\frac{\|T(1_{A_4}+\alpha 1_{A_3} + \epsilon 1_{A_1})\|^p - \|T(1_{A_4}+\alpha 1_{A_3})\|^p}{\epsilon}\\
&\qquad - \lim_{\epsilon\uparrow 0}\frac{\|T(1_{A_4}+\alpha 1_{A_3} + \epsilon 1_{A_1})\|^p - \|T(1_{A_4}+\alpha 1_{A_3})\|^p}{\epsilon}\\
&= p\int_{\beta_\alpha}^{\beta_\alpha+\gamma\lambda} s^p w(t)dt - p\int_{\beta_\alpha+\lambda(\mu(A_4)-\gamma)}^{\beta_\alpha+\lambda\mu(A_4)} s^p w(t)dt.
\end{aligned}$$

Differentiating both sides with respect to γ, we have

$$w(\gamma) - w(\mu(A_4) - \gamma) = s^p\lambda w(\beta_\alpha + \gamma\lambda) - s^p\lambda w(\beta_\alpha + (\mu(A_4) - \gamma)\lambda).$$

Proof of claim (b): Let f_n be a sequence of measurable functions which converges to f pointwise, and let c be any real number. It is known that

$$\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{t\colon f_n(t) > c - \frac{1}{n}\} \subseteq \{t\colon f(t) \ge c\};$$
$$\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{t\colon f_n(t) < c + \frac{1}{n}\} \subseteq \{t\colon f(t) \le c\}.$$

Let α_n be a sequence in $(0,1)$ which converges to α. We will show that

$$\limsup_{n\to\infty}\beta_{\alpha_n} \le \beta_\alpha \le \liminf_{n\to\infty}\beta_{\alpha_n}.$$

Suppose it is not true. By passing to a subsequence, we may assume that $\lim_{n\to\infty}\beta_{\alpha_n}$ exists, $\lim_{n\to\infty}\beta_{\alpha_n} \neq \beta_\alpha$, and $|T(1_{A_4} + \alpha_n 1_{A_3})|$ converges to $|T(1_{A_4} + \alpha 1_{A_3})|$ a.e. Note:

(iv) if $0 \le \alpha < 1$, then $\beta_\alpha + \lambda\mu(A_4) = \mu(|T(1_{A_4} + \alpha 1_{A_3}) \ge s)$;

(v) $\beta_1 + \lambda(\mu(A_3 \cup A_4)) = \mu(|T\,1_{A_4 \cup A_3}| \ge s)$;

(vi) $1 - \beta_\alpha = \mu(|T(1_{A_4} + \alpha 1_{A_3})| \le s)$.

We have

$$\limsup_{n\to\infty}\beta_{\alpha_n} + \lambda\mu(A_4) \le \beta_\alpha + \lambda\mu(A_4) \quad \text{when } \alpha < 1,$$
$$\limsup_{n\to\infty}(1-\beta_{\alpha_n}) \le 1 - \beta_\alpha.$$

Hence,

$$\limsup_{n\to\infty}\beta_{\alpha_n} \le \beta_\alpha \qquad \text{when } \alpha < 1,$$
$$\liminf_{n\to\infty}\beta_{\alpha_n} \ge \beta_\alpha.$$

If $\alpha = 1$, then

$$\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{t\colon |T(1_{A_4} + \alpha_n 1_{A_3})|(t) > s - 2(1-\alpha_n)s\} \subseteq \{t\colon |T(1_{A_4} + 1_{A_3})|(t) \ge s\}$$

and

$$\limsup_{n\to\infty}\beta_{\alpha_n} + \lambda(\mu(A_4) + \mu(A_3)) \le \beta_1 + \lambda(\mu(A_4) + \mu(A_3)).$$

So we have $\lim_{n\to\infty}\beta_{\alpha_n} = \beta_\alpha$. We proved claim (b).

We claim that $\beta_1 = 0$. Suppose it is not true. We note that:

(vii) if γ is fixed, the lefthand side of formula in claim (a) is a constant;

(viii) $[\beta_0, \beta_1] \subseteq \{\beta_\alpha\colon\ 0 \le \alpha \le 1\}$;

(ix) $\beta_1 = \mu(|T\,1_A| > s) > 0$ and $\beta_0 = \mu(|T\,1_{A_1 \cup A_2}| > s) = \mu(|T\,1_{A_4}| > s)$.

For any $\gamma < \frac{\mu(A_4)}{2}$ and for almost all $\eta \in [\beta_0, \beta_1]$,

$$\begin{aligned} w(\gamma) - w(\gamma + (\mu(A_4) - 2\gamma)) &= w(\gamma) - w(\mu(A_4) - \gamma) \\ &= s^p\lambda w(\eta + \gamma\lambda) - s^p\lambda w(\eta + (\mu(A_4) - \gamma)\lambda) \\ &= s^p\lambda(w(\eta + \gamma\lambda) - w((\eta + \gamma\lambda) + (\mu(A_4) - 2\gamma)\lambda)). \end{aligned}$$

This implies w' is constant on $[\beta_0, \beta_1]$. (So w is linear on $[\beta_0, \beta_1]$.) Note:

(x) if $A = A_1 \cup A_2 \cup A_3$ is fixed, then β_1 is fixed;

(xi) $\lim_{\mu(B)\to 0}\mu(|T\,1_B| > s) = 0$ (since T is continuous).

Fix the set A and let $\mu(A_4)$ tend to zero. Then β_0 tends to zero. This implies that w must be linear on $(0,\beta_1)$. By (11), $s = 1$. We get a contradiction. So $\beta_1 = 0$.

(13) We claim that for any measurable set A, $\mu(0 < |T1_A| < s) = 0$. Suppose it is not true (so $\mu(0 < |T1_A| < s) > 0$ for some measurable set A). Let $\{B_1, B_2, \ldots, B_k\}$ be any partition of A. Since $1_A = \sum_{j=1}^k 1_{B_j}$, there is j' such that $T1_{B_{j'}}|_{\{t:\, 0<|T1_A|(t)<s\}} \neq 0$. But $\mu(|T1_{B_{j'}}| > s) = 0$ and $\{t\colon |T1_{B_{j'}}|(t) = s\} \subseteq \{t\colon |T1_A|(t) = s\}$. So we must have $\mu(0 < |T1_{B_{j'}}| < s) > 0$ for some j', $1 \le j' \le k$. So we may assume that $\mu(A) < \frac{1}{2}$. Let A_1 and A_2 two disjoint measurable subsets of $(0,1) \setminus A$. Note:

(i) $\mu(|T1_B| > s) = 0$ for any measurable subset B of $(0,1)$;

(ii) $\mu(|T(1_A + \alpha 1_{A_1\cup A_2})| = \alpha s) = \lambda\mu(A_1 \cup A_2)$ for $0 < \alpha < 1$.

As the proof of (12) claim (b),

(iii) the mapping $\alpha \to \beta_\alpha = \mu(|T(1_A + \alpha 1_{A_1\cup A_2})| > \alpha s)$ is continuous on $(0,1)$;

(iv) $\lim\limits_{\alpha\uparrow 1} \mu(|T(1_A + \alpha 1_{A_1\cup A_2})| > \alpha s) = \lambda\mu(A)$;

(v) $\limsup\limits_{\alpha\downarrow 0} \mu(|T(1_A + \alpha 1_{A_1\cup A_2})| > 0) \ge \mu(|T1_A| > 0) = \beta_0$. (Note: we do not know whether $\mu(|T(1_A + \alpha 1_{A_1\cup A_2})| = 0) = \mu(|T1_A| = 0)$ for all $0 < \alpha < 1$.)

Let $\gamma = \mu(A_1)$. Compute

$$\lim_{\epsilon\downarrow 0} \frac{\|(1_A + \alpha 1_{A_1\cup A_2}) + \epsilon 1_{A_1}\|^p - \|1_A + \alpha 1_{A_1\cup A_2}\|^p}{\epsilon} - \lim_{\epsilon\uparrow 0} \frac{\|(1_A + \alpha 1_{A_1\cup A_2}) + \epsilon 1_{A_1}\|^p - \|1_A + \alpha 1_{A_1\cup A_2}\|^p}{\epsilon}$$

as (12). We have

$$\begin{aligned}&\int_{\mu(A)}^{\mu(A)+\gamma} \alpha^{p-1} w(t)dt - \int_{\mu(A)+(\mu(A_1\cup A_2)-\gamma)}^{\mu(A)+\mu(A_1\cup A_2)} \alpha^{p-1} w(t)dt \\ &= \int_{\beta_\alpha}^{\beta_\alpha+\gamma\lambda} s^p\alpha^{p-1} w(t)dt - \int_{\beta_\alpha+\lambda(\mu(A_1\cup A_2)-\gamma)}^{\beta_\alpha+\lambda\mu(A_1\cup A_2)} s^p\alpha^{p-1} w(t)dt\end{aligned}$$

and

$$\begin{aligned}&w(\mu(A)+\gamma) - w(\mu(A)+\mu(A_1\cup A_2)-\gamma) \\ &= s^p\lambda[w(\beta_\alpha+\gamma\lambda) - w(\beta_\alpha+\lambda(\mu(A_1\cup A_2)-\gamma))].\end{aligned}$$

Let $A_1 \cup A_2$ be a fixed measurable subset of $(0,1)$, and let $\beta_1 = \lim_{\alpha\uparrow 1} \beta_\alpha = \lambda\mu(A)$. Since $\mu(0 < |T 1_A| < s) > 0$, $\beta_1 < \beta_0$. As the proof of (12), we have that w is linear on $(\beta_1 + \frac{\lambda\mu(A_1\cup A_2)}{2}, \beta_0 + \frac{\lambda\mu(A_1\cup A_2)}{2})$.

Note: β_0 and β_1 depend only on A. Hence, for any $\eta < \frac{1}{2}$, there exist two disjoint measurable subsets A_1, A_2 of $(0,1)\setminus A$ such that $\mu(A_1 \cup A_2) = \eta$. So w is linear on $(\beta_1 + \frac{\lambda\eta}{2}, \beta_0 + \frac{\lambda\eta}{2})$ for any $0 < \eta < \frac{1}{2}$. This implies w is linear on $(\beta_1, \beta_1 + \frac{\lambda}{4})$.

Now, let $\mu(A)$ tend to zero. Then β_1 tends to zero. This implies w is linear on $(0, \frac{\lambda}{4})$. By (11), $s = 1$, and we get a contradiction. So we proved the general case.

5. The isometries on complex $L_{w,p}$

In the complex $L_{w,p}$, Lemma 1, Lemma 7, Corollary 6 and Remark 3 become

LEMMA 1′: *Suppose that $1 < p < \infty$. Let f be a positive decreasing function in $L_{w,p}$ and let g be any function in $L_{w,p}$ such that if $\mu(f = s) \neq 0$, then $\Re g|_{f^{-1}(s)}$ is decreasing (respectively, increasing). Then*

$$\lim_{\epsilon\downarrow 0} \frac{\|f+\epsilon g\|^p - \|f\|^p}{\epsilon} = p\int (\Re g) f^{p-1}\, w dt$$

$$(\textit{respectively, } \lim_{\epsilon\uparrow 0} \frac{\|f+\epsilon g\|^p - \|f\|^p}{\epsilon} = p\int (\Re g) f^{p-1}\, w dt).$$

LEMMA 7′: *Let f be a positive decreasing function in $L_{w,1}$ and let g be any function in $L_{w,1}$ such that*

(i) *$|g|$ is decreasing on $\{t : f(t) = 0\}$;*
(ii) *if $s > 0$ and $\mu(f = s) \neq 0$, then $\Re g|_{f^{-1}(s)}$ is decreasing (respectively, increasing).*

Then

$$\lim_{\epsilon\downarrow 0} \frac{\|f+\epsilon g\| - \|f\|}{\epsilon} = \int_{\operatorname{supp} f} (\Re g)\, w dt + \int_{\mathbb{R}\setminus \operatorname{supp} \mho} |g|\, w dt$$

$$(\textit{respectively, } \lim_{\epsilon\uparrow 0} \frac{\|f+\epsilon g\| - \|f\|}{\epsilon} = \int_{\operatorname{supp} f} (\Re g)\, w dt + \int_{\mathbb{R}\setminus \operatorname{supp} \mho} |g|\, w dt).$$

COROLLARY 6′: *Let T be an isometry on $L_{w,p}$ and let A, B be two disjoint finite measurable sets. Let $f = T 1_A$ and $g = T 1_B$. Then for any complex number α, the following are equivalent.*

(a) *there exists $s > 0$ and a such that $|a| = 1$, $\mu(|f + \alpha g| = s) > 0$ and the restriction of $\Re\,(a\, g \operatorname{sgn} \overline{f + \alpha g})$ to $|f + \alpha g|^{-1}(s)$ is not constant;*

(b) *there exists a such that* $|a| = 1$ *and*

$$\lim_{\epsilon\downarrow 0}\frac{\|(f+\alpha g)+a\,\epsilon g\|^p-\|f+\alpha g\|^p}{\epsilon}\neq\lim_{\epsilon\uparrow 0}\frac{\|(f+\alpha g)+a\,\epsilon g\|^p-\|f+\alpha g\|^p}{\epsilon};$$

(c) *there exists a such that* $|a| = 1$ *and*

$$\lim_{\epsilon\downarrow 0}\frac{\|(1_A+\alpha 1_B)+a\,\epsilon 1_B\|^p-\|1_A+\alpha 1_B\|^p}{\epsilon}\neq\lim_{\epsilon\uparrow 0}\frac{\|(1_A+\alpha 1_B)+a\,\epsilon 1_B\|^p-\|1_A+\alpha 1_B\|^p}{\epsilon};$$

(d) $|\alpha| = 1$.

Remark 3′: Let f and g be as above. Suppose that $s > 0$ and $\mu(|f| = s) > 0$. Then both $\Re(g\,\mathrm{sgn}\,\bar f)$ and $\Re(ig\cdot\mathrm{sgn}\,\bar f)$ are constant on $|f|^{-1}(s)$. Hence, there exists a such that

(i) $|a| = 1$ and $\mathrm{sgn} f\,|_{|f|^{-1}(s)} = a\,\mathrm{sgn}\,g\,|_{|f|^{-1}(s)}$;
(ii) $|g|$ is constant on $|f|^{-1}(s)$. ∎

One can easily verify that (1)–(3) are still true in complex $L_{w,p}$.

(14) Let A_1 be a measurable subset of A and let $f_1 = T1_{A_1}$, $f_2 = T1_{A\setminus A_1}$ and $f = T1_A$. We claim that of $T1_{A_1}\,\mathrm{sgn}\,f$ is real on the set $|f|^{-1}(s)$.

Without of loss of generality, we may assume that f is a nonnegative decreasing function and $\mu(A\setminus A_1)\neq 0$. Since

$$\lim_{\epsilon\downarrow 0}\frac{\|1_A+i\,\epsilon 1_{A_1}\|^p-\|1_A\|^p}{\epsilon}=0=\lim_{\epsilon\uparrow 0}\frac{\|1_A+i\,\epsilon 1_{A_1}\|^p-\|1_A\|^p}{\epsilon},$$

the restriction of $\mathrm{Im} f_1$ to $f^{-1}(s)$ is constant. But $\mu(|f_2| = s) > 0$ and $|f_2|^{-1}(s)\subseteq |f|^{-1}(s)$. So $f_1|_{|f_2|^{-1}(s)} = 0$, and the restriction of $\mathrm{Im} f_1$ to $f^{-1}(s)$ must be zero. We proved our claim.

One can easily verify (4)–(13). So the Main Theorem is still true in complex $L_{w,p}$.

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